On the tractability of some natural packing, covering and partitioning problems

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July 21, 2014

Abstract

In this paper we fix 7 types of undirected graphs: paths, paths with prescribed endvertices, circuits, forests, spanning trees, (not necessarily spanning) trees and cuts. Given an undirected graph $G = (V, E)$ and two "object types" A and B chosen from the alternatives above, we consider the following questions. **Packing problem:** can we find an object of type A and one of type B in the edge set E of G, so that they are edge-disjoint? **Partitioning problem:** can we partition E into an object of type A and one of type B? **Covering problem:** can we cover E with an object of type A, and an object of type B? This framework includes 44 natural graph theoretic questions. Some of these problems were well-known before, for example covering the edge-set of a graph with two spanning trees, or finding an s-t path P and an s'-t' path P' that are edge-disjoint. However, many others were not, for example can we find an s-t path $P \subseteq E$ and a spanning tree $T \subseteq E$ that are edge-disjoint? Most of these previously unknown problems turned out to be NP-complete, many of them even in planar graphs. This paper determines the status of these 44 problems. For the NP-complete problems we also investigate the planar version, for the polynomial problems we consider the matroidal generalization (wherever this makes sense).

1 Introduction

In this paper we consider undirected graphs. The node set of a graph $G = (V, E)$ is sometimes also denoted by $V(G)$, and similarly, the edge set is sometimes denoted by $E(G)$. A **subgraph** of a graph $G = (V, E)$ is a pair (V', E') where $V' \subseteq V$ and $E' \subseteq E \cap (V' \times V')$. A graph is called **subcubic** if every node is incident to at most 3 edges, and it is called **subquadratic** if every node is incident to at most 4 edges. By a **cut** in a graph we mean the set of edges leaving a nonempty proper subset V' of the nodes (note that we do not require that V' and $V - V'$ induces a connected graph). We use standard terminology and refer the reader to [9] for what is not defined here.

We consider 3 types of decision problems with 7 types of objects. The three types of problems are: packing, covering and partitioning, and the seven types of objects are the following: paths (denoted by a P), paths with specified endvertices (denoted by P_{st}, where s and t are the prescribed endvertices), (simple) circuits (denoted by C: by that we mean a closed walk of length at least 2, without edge- and node-repetition), forests (F), spanning trees (SpT), (not necessarily spanning) trees (T), and cuts (denoted by Cut). Let $G = (V, E)$ be a **connected** undirected graph (we assume connectedness in order to avoid trivial case-checkings) and A and B two (not necessarily different) object types from the 7 possibilities above. The general questions we ask are the following:

- **Packing problem** (denoted by A$\wedge$B): can we **find two edge-disjoint subgraphs** in G, one of type A and the other of type B?
- **Covering problem** (denoted by A$\cup$B): can we **cover the edge set** of G with an object of type A and an object of type B?
- **Partitioning problem** (denoted by A+B): can we **partition the edge set** of G into an object of type A and an object of type B?

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Let us give one example of each type. A typical partitioning problem is the following: decide whether the edge set of G can be partitioned into a spanning tree and a forest. Using our notations this is Problem SpT+F. This problem is in **NP** $\cap$ **co-NP** by the results of Nash-Williams [18], polynomial algorithms for deciding the problem were given by Kishi and Kajitani [15], and Kameda and Toida [13].

A typical packing problem is the following: given four (not necessarily distinct) vertices $s, t, s', t' \in V$, decide whether there exists an s-t path P and an s'-t'-path P' in G, such that P and P' do not share any edge. With our notations this is Problem $\mathrm{P}_{st} \wedge \mathrm{P}_{s't'}$. This problem is still solvable in polynomial time, as was shown by Thomassen [24] and Seymour [23].

A typical covering problem is the following: decide whether the edge set of G can be covered by a path and a circuit. In our notations this is Problem $P \cup C$. Interestingly we found that this simple-looking problem is NP-complete.

Let us introduce the following short formulation for the partitioning and covering problems. If the edge set of a graph G can be partitioned into a type A subgraph and a type B subgraph then we will also say that **the edge set of G is $A+B$**. Similarly, if there is a solution of Problem $A \cup B$ for a graph G then we say that **the edge set of G is $A \cup B$**.

Table 1: 25 PARTITIONING PROBLEMS

Problem	**Status**	**Reference**	**Remark**
P+P	**NPC**	Theorem 5	**NPC**for subquadratic planar
P+P_{st}	**NPC**	Theorem 5	**NPC**for subquadratic planar
P+C	**NPC**	Theorem 5	**NPC**for subquadratic planar
P+T	**NPC**	Theorem 5	**NPC**for subquadratic planar
P+SpT	**NPC**	Theorem 5	**NPC**for subquadratic planar
P+F	**NPC**	Theorem 3 (and Theorem 5)	**NPC**for subcubic planar
P_{st}+$\mathrm{P}_{s't'}$	**NPC**	Theorem 5	**NPC**for subquadratic planar
P_{st}+C	**NPC**	Theorem 5	**NPC**for subquadratic planar
P_{st}+T	**NPC**	Theorem 5	**NPC**for subquadratic planar
P_{st}+SpT	**NPC**	Theorem 5	**NPC**for subquadratic planar
P_{st}+F	**NPC**	Theorem 3 (and Theorem 5)	**NPC**for subcubic planar
C+C	**NPC**	Theorem 5	**NPC**for subquadratic planar
C+T	**NPC**	Theorem 5	**NPC**for subquadratic planar
C+SpT	**NPC**	Theorem 5	**NPC**for subquadratic planar
C+F	**NPC**	Theorem 3 (and Theorem 5)	**NPC**for subcubic planar
T+T	**NPC**	Pálvölgyi [20]	planar graphs?
T+SpT	**NPC**	Theorem 6	planar graphs?
F+F	**P**	Kishi and Kajitani [15], Kameda and Toida [13] (Nash-Williams [18])	in **P**for matroids: Edmonds [7]
SpT+SpT	**P**	Kishi and Kajitani [15], Kameda and Toida [13], (Nash-Williams [19], Tutte [25])	in **P**for matroids: Edmonds [7]
Cut+Cut	**P**	if and only if bipartite (and $\lvert V\rvert \geq 3$)	
Cut+F	**NPC**	Theorem 7	planar graps?
Cut+C	**NPC**	Theorem 3	**NPC**for subcubic planar
Cut+T	**NPC**	Theorem 3	**NPC**for subcubic planar
Cut+P	**NPC**	Theorem 3	**NPC**for subcubic planar
Cut+P_{st}	**NPC**	Theorem 3	**NPC**for subcubic planar

The setting outlined above gives us 84 problems. Note however that some of these can be omitted. For example $\mathrm{P} \wedge \mathrm{A}$ is trivial for each possible type A in question, because P may consist of only one vertex. By the same reason, $\mathrm{T} \wedge \mathrm{A}$ and $\mathrm{F} \wedge \mathrm{A}$ type problems are also trivial. Furthermore, observe that the edge-set $E(G)$ of a graph G is $\mathrm{F}+\mathrm{A} \Leftrightarrow E(G)$ is $\mathrm{F} \cup \mathrm{A} \Leftrightarrow E(G)$ is $\mathrm{T} \cup \mathrm{A} \Leftrightarrow E(G)$ is $\mathrm{SpT} \cup \mathrm{A}$: therefore we will only consider the problems of form F+A among these for any A. Similarly, the edge set $E(G)$ is $\mathrm{F}+\mathrm{F} \Leftrightarrow E(G)$ is $\mathrm{T}+\mathrm{F} \Leftrightarrow E(G)$ is SpT+F: again we choose to deal with F+F. We can also omit the problems Cut+SpT and $\mathrm{Cut} \wedge \mathrm{SpT}$ because a cut and a spanning tree can never be disjoint.

The careful calculation gives that we are left with 44 problems. We have investigated the status of these. Interestingly, many of these problems turn out to be NP-complete. Our results are summarized in Tables 1-3. We note that in our NP-completeness proofs we always show that the considered problem is NP-complete even if the input graph is simple. On the other hand, the polynomial algorithms given here always work also for multigraphs (we allow parallel edges, but we forbid loops). Some of the results shown in the tables were already proved in the preliminary version [5] of this paper: namely we have already shown the NP-completeness of Problems P+T, P+SpT, P_{st}+T, P_{st}+SpT, C+T, C+SpT, T+SpT, $P_{st}\wedge$SpT, and C$\wedge$SpT there.

Table 2: 9 PACKING PROBLEMS

Problem	**Status**	**Reference**	**Remark**
$P_{st}\wedge P_{s't'}$	**P**	Seymour [23], Thomassen [24]	
$P_{st}\wedge C$	**P**	see Section 3	
$P_{st}\wedge$SpT	**NPC**	Theorem 6	planar graphs?
$C\wedge C$	**P**	Bodlaender [6] (see also Section 3)	**NPC**in linear matroids (Theorem 10)
C$\wedge$SpT	**NPC**	Theorem 6	polynomial in planar graphs, [4]
SpT$\wedge$SpT	**P**	Imai [12], (Nash-Williams [19], Tutte [25])	in **P**for matroids: Edmonds [7]
Cut$\wedge$Cut	**P**	always, if G has two non-adjacent vertices	**NPC**in linear matroids (Corollary 12)
Cut$\wedge P_{st}$	**P**	always, except if the graph is an s-t path (with multiple copies for some edges)	
Cut$\wedge$C	**P**	always, except if the graph is a tree, a circuit, or a bunch of parallel edges	**NPC**in linear matroids ($\Leftrightarrow$ the matroid is not uniform, Theorem 9)

Table 3: 10 COVERING PROBLEMS

Problem	**Status**	**Reference**	**Remark**
$P\cup P$	**NPC**	Theorem 5	**NPC**for subquadratic planar
$P\cup P_{st}$	**NPC**	Theorem 5	**NPC**for subquadratic planar
$P\cup C$	**NPC**	Theorem 5	**NPC**for subquadratic planar
$P_{st}\cup P_{s't'}$	**NPC**	Theorem 5	**NPC**for subquadratic planar
$P_{st}\cup C$	**NPC**	Theorem 5	**NPC**for subquadratic planar
$C\cup$C	**NPC**	Theorem 5	**NPC**for subquadratic planar
Cut$\cup$Cut	**NPC**	if and only if 4-colourable	always in planar Appel et al. [1], [2]
Cut$\cup$C	**NPC**	Theorem 3	**NPC**for subcubic planar
Cut$\cup$P	**NPC**	Theorem 3	**NPC**for subcubic planar
Cut$\cup P_{st}$	**NPC**	Theorem 3	**NPC**for subcubic planar

Problems P_{st}+SpT and T+SpT were posed in the open problem portal called "EGRES Open" [8] of the Egerváry Research Group. Most of the NP-complete problems remain NP-complete for planar graphs, though we do not know yet the status of Problems T+T, T+SpT, Cut+F $P_{st}\wedge$SpT, and C$\wedge$SpT for planar graphs, as indicated in the table.

We point out to an interesting phenomenon: planar duality and the NP-completeness of Problem C+C gives that deciding whether the edge set of a planar graph is the disjoint union of two *simple* cuts is NP-complete (a **simple cut**, or **bond** of a graph is an inclusionwise minimal cut). In contrast, the edge set of a graph is Cut+Cut if and only if the graph is bipartite on at least 3 nodes[1], that is

[1]It is easy to see that the edge set of a connected bipartite graph on at least 3 nodes is Cut+Cut. On the other hand, the intersection of a cut and a circuit contains an even number of edges, therefore the edge set of a non-bipartite graph cannot be Cut+Cut.

Cut+Cut is polinomially solvable even for non-planar graphs.

Some of the problems can be formulated as a problem in the graphic matroid and therefore also have a natural matroidal generalization. For example the matroidal generalization of C∧C is the following: can we find two disjoint circuits in a matroid (given with an independence oracle, say)? Of course, such a matroidal question is only interesting here if it can be solved for graphic matroids in polynomial time. Some of these matroidal questions is known to be solvable (e.g., the matroidal version of SpT+SpT), and some of them was unknown (at least for us): the best example being the (above mentioned) matroidal version of C∧C. In the table above we indicate these matroidal generalizations, too, where the meaning of the problem is understandable. The matroidal generalization of spanning trees, forests, circuits is straightforward. We do not want to make sense of trees, paths, or s-t-paths in matroids. On the other hand, cuts deserve some explanation. In matroid theory, a **cut** (also called **bond** in the literature) of a matroid is defined as an inclusionwise minimal subset of elements that intersects every base. In the graphic matroid this corresponds to a simple cut of the graph defined above. So we will only consider packing problems for cuts in matroids: for example the problem of type A∧Cut in graphs is equivalent to the problem of packing A and a simple cut in the graph, therefore the matroidal generalization is understandable. We will discuss these matroidal generalizations in Section 4.

2 NP-completeness proofs

A graph $G = (V, E)$ is said to be **subcubic** if $d_G(v) \leq 3$ for every $v \in V$. In many proofs below we will use Problem Planar3RegHam and Problem Planar3RegHam$-e$ given below.

Problem 1 (Planar3RegHam)**.** *Given a* 3*-regular planar graph* $G = (V, E)$*, decide whether there is a Hamiltonian circuit in* G*.*

Problem 2 (Planar3RegHam$-e$)**.** *Given a* 3*-regular planar graph* $G = (V, E)$ *and an edge* $e \in E$*, decide whether there is a Hamiltonian circuit in* G *through edge* e*.*

It is well-known that Problems Planar3RegHam and Planar3RegHam$-e$ are NP-complete (see Problem [GT37] in [10]).

2.1 NP-completeness proofs for subcubic planar graphs

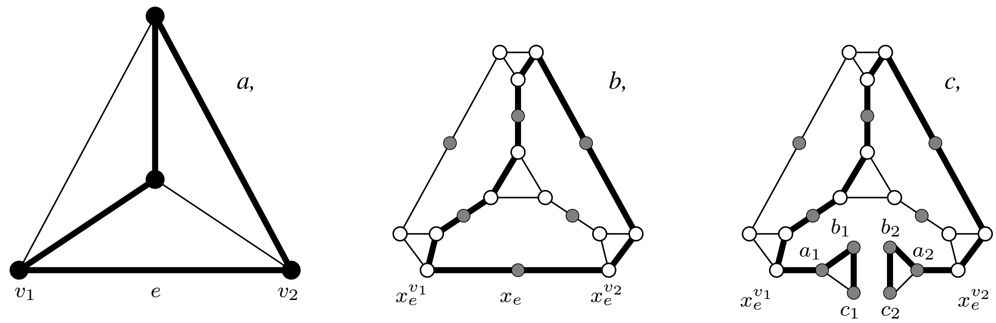

Figure 1: An illustration for the proof of Theorem 3.

Theorem 3. *The following problems are NP-complete, even if restricted to subcubic planar graphs:* Cut∪C*,* Cut+C*,* C+F*,* Cut∪P*,* Cut∪P$_{st}$*,* Cut+P*,* Cut+P$_{st}$*,* Cut+T*,* P+F*,* P$_{st}$+F.

Proof. All the problems are clearly in NP. First we prove the completeness of Cut∪C, Cut+C and C+F using a reduction from Problem Planar3RegHam. Given an instance of the Problem Planar3RegHam with the 3-regular planar graph G, construct the following graph G'. First subdivide each edge $e = v_1v_2 \in E(G)$ with 3 new nodes $x_e^{v_1}, x_e, x_e^{v_2}$ such that they form a path in the order $v_1, x_e^{v_1}, x_e, x_e^{v_2}, v_2$. Now for any node $u \in V(G)$ and any pair of edges $e, f \in E(G)$ incident to u connect x_e^u and x_f^u with a new edge. Finally, delete all the original nodes $v \in V(G)$ to get G'. Informally speaking, G' is obtained from G by blowing a triangle into every node of G and subdividing each original edge with a new node: see Figure 1 *a,b,* for an illustration. Note that by contracting these triangles in G' and undoing the subdivision vertices of form x_e gives back G. Clearly, the resulting graph G' is still planar and has maximum degree 3 (we mention that the subdivision nodes of form x_e are only needed for the Problem Cut+C). We will prove that G contains a Hamiltonian circuit if and only if G' contains a circuit covering odd circuits (i.e., the edge-set of G' is C∪Cut) if and only if the edge-set of G' is C+Cut if and only if G' contains a circuit covering all the circuits (i.e., the edge set of G' is C+F). First let C be a Hamiltonian circuit in G. We define a circuit C' in G' as follows. For any $v \in V(G)$, if C uses the edges e, f incident to v then let C' use the 3 edges $x_e x_e^v, x_e^v x_f^v, x_f^v x_f$ (see Figure 1 *a,b,* for an illustration). Observe that $G' - C'$ is a forest, proving that the edge-set of G' is C+F. Similarly, the edge set of $G' - C'$ is a cut of G', proving that the edge-set of G' is C+Cut. Finally we show that if the edge set of G' is C∪F then G contains a Hamiltonian circuit: this proves the sequence of equivalences stated above (the remaining implications being trivial). Assume that G' has a circuit C' that intersects the edge set of every odd circuit of G'. Contract the triangles of G' and undo the subdivision nodes of form x_e and observe that C' becomes a Hamiltonian circuit of G.

For the rest of the problems we use Planar3RegHam$-e$. Given the 3-regular planar graph G and an edge $e = v_1v_2 \in E(G)$, first construct the graph G' as above. Next modify G' the following way: if $x_e^{v_1}, x_e, x_e^{v_2}$ are the nodes of G' arising from the subdivision of the original edge $e \in E(G)$ then let $G'' = (G' - x_e) + \{x_e^{v_i} a_i, a_i b_i, b_i c_i, c_i a_i : i = 1, 2\}$, where $a_i, b_i, c_i (i = 1, 2)$ are 6 new nodes (informally, "cut off" the path $x_e^{v_1}, x_e, x_e^{v_2}$ at x_e and substitute the arising two vertices of degree 1 with two triangles). An illustration can be seen in Figure 1 *a,c.*

Let $s = c_1$ and $t = c_2$. The following chain of equivalences settles the NP-completeness of the rest of the problems promised in the theorem. The proof is similar to the one above and is left to the reader.

There exists a Hamiltonian circuit in G using the edge $e \Leftrightarrow$ the edge set of G'' is Cut+P_{st} $\Leftrightarrow$ the edge set of G'' is Cut+P $\Leftrightarrow$ the edge set of G'' is Cut+T $\Leftrightarrow$ the edge set of G'' is Cut∪P_{st} $\Leftrightarrow$ the edge set of G'' is Cut∪P $\Leftrightarrow$ the edge set of G'' is P_{st}+F $\Leftrightarrow$ the edge set of G'' is P+F. □

2.2 NP-completeness proofs based on Kotzig's theorem

Now we prove the NP-completeness of many other problems in our collection using the following elegant result proved by Kotzig [16].

Theorem 4. *A 3-regular graph contains a Hamiltonian circuit if and only if the edge set of its line graph can be decomposed into two Hamiltonian circuits.*

This theorem was used to prove NP-completeness results by Pike in [22]. Another useful and well known observation is the following: the line graph of a planar 3-regular graph is 4-regular and planar.

Theorem 5. *The following problems are NP-complete, even if restricted to subquadratic planar graphs:* P+P, P+P_{st}, P+C, P+T, P+SpT, P+F, P_{st}+$\mathrm{P}_{s't'}$, P_{st}+C, P_{st}+F, P_{st}+T, P_{st}+SpT, C+C, C+T, C+SpT, C+F, P∪P, P∪P_{st}, P∪C, P_{st}∪$\mathrm{P}_{s't'}$, P_{st}∪C, C∪C.

Proof. The problems are clearly in NP. Let G be a planar 3-regular graph. Since $L(G)$ is 4-regular, it is decomposable to two circuits if and only if it is decomposable to two Hamiltonian circuits. This together with Kotzig's theorem shows that C+C is NP-complete. For every other problem of type C+A use $L = L(G) - st$ with an arbitrary edge st of $L(G)$. Let C be a circuit of L and observe that (by the number of edges of L and the degree conditions) $L - C$ is circuit-free if and only if C is a Hamiltonian circuit and $L - C$ is a Hamiltonian path connecting s and t.

For the rest of the partitioning type problems we need one more trick. Let us be given a 3-regular planar graph $G = (V, E)$ and an edge $e = xy \in E$. We construct another 3-regular planar graph $G' = (V', E')$ as follows. Delete edge xy, add vertices x', y', and add edges xx', yy' and add two parallel edges between x' and y', namely e_{xy} and f_{xy} (note that G' is planar, too). Clearly G has a Hamiltonian circuit through edge xy if and only if G' has a Hamiltonian circuit. Now consider $L(G')$, the line graph of G', it is a 4-regular planar graph. Note, that in $L(G')$ there are two parallel edges between nodes $s = e_{xy}$ and $t = f_{xy}$, call these edges g_1 and g_2. Clearly, $L(G')$ can be decomposed into two Hamiltonian circuits if and only if $L' = L(G') - g_1 - g_2$ can be decomposed into two Hamiltonian paths. Let P be a path in L' and notice again that the number of edges of L' and the degrees of the nodes in L' imply that $L' - P$ is circuit free if and only if P and $L' - P$ are two Hamiltonian paths in L'.

Finally, the NP-completeness of the problems of type A∪B is an easy consequence of the NP-completeness of the corresponding partitioning problem A+B: use the same construction and observe that the number of edges enforce the two objects in the cover to be disjoint. □

We remark that the above theorem gives a new proof of the NP-completeness of Problems C+F, P+F and P_{st}+F, already proved in Theorem 3.

2.3 NP-completeness of Problems P_{st}∧SpT, T+SpT, C∧SpT, and Cut+F

First we show the NP-completeness of Problems P_{st}∧SpT, T+SpT, and C∧SpT. Problem T+T was proved to be NP-complete by Pálvölgyi in [20] (the NP-completeness of this problem with the additional requirement that the two trees have to be of equal size was proved by Pferschy, Woeginger and Yao [21]). Our reductions here are similar to the one used by Pálvölgyi in [20]. We remark that our first proof for the NP-completeness of Problems P+T, P+SpT, P_{st}+T, P_{st}+SpT, C+T and C+SpT used a variant of the construction below (this can be found in [5]), but later we found that using Kotzig's result (Theorem 4) a simpler proof can be given for these.

For a subset of edges $E' \subseteq E$ in a graph $G = (V, E)$, let $V(E')$ denote the subset of nodes incident to the edges of E', i.e., $V(E') = \{v \in V : \text{there exists an } f \in E' \text{ with } v \in f\}$.

Theorem 6. *Problems* $\mathrm{P}_{st} \wedge \mathrm{SpT}$, T+SpT *and* $\mathrm{C} \wedge \mathrm{SpT}$ *are NP-complete even for graphs with maximum degree 3.*

Proof. It is clear that the problems are in NP. Their completeness will be shown by a reduction from the well known NP-complete problems 3SAT or the problem One-In-Three 3SAT (Problems LO2 and LO4 in [10]). Let φ be a 3-CNF formula with variable set $\{x_1, x_2, \ldots, x_n\}$ and clause set $\mathcal{C} = \{C_1, C_2, \ldots, C_m\}$ (where each clause contains exactly 3 literals). Assume that literal x_j appears in k_j clauses $C_{a_1^j}, C_{a_2^j}, \ldots, C_{a_{k_j}^j}$, and literal $\overline{x_j}$ occurs in l_j clauses $C_{b_1^j}, C_{b_2^j}, \ldots, C_{b_{l_j}^j}$. Construct the following graph $G_\varphi = (V, E)$.

For an arbitrary clause $C \in \mathcal{C}$ we will introduce a new node u_C, and for every literal y in C we introduce two more nodes $v(y, C), w(y, C)$. Introduce the edges $u_C w(y, C), w(y, C)v(y, C)$ for every clause C and every literal y in C (the nodes $w(y, C)$ will have degree 2).

For every variable x_j introduce 8 new nodes z_1^j, z_2^j, $w_1^j, \overline{w_1^j}$, w_2^j, $\overline{w_2^j}$, $w_3^j, \overline{w_3^j}$. For every variable x_j, let G_φ contain a circuit on the $k_j + l_j + 4$ nodes z_1^j, $v(x_j, C_{a_1^j})$, $v(x_j, C_{a_2^j})$, $\ldots$, $v(x_j, C_{a_{k_j}^j})$, w_1^j, z_2^j, $\overline{w_1^j}$, $v(\overline{x_j}, C_{b_{l_j}^j})$, $v(\overline{x_j}, C_{b_{l_j-1}^j})$, $\ldots$, $v(\overline{x_j}, C_{b_1^j})$ in this order. We say that this circuit is **associated to variable** x_j. Connect the nodes z_2^j and z_1^{j+1} with an edge for every $j = 1, 2, \ldots, n-1$. Introduce furthermore a path on nodes $w_3^1, \overline{w_3^1}, w_3^2, \overline{w_3^2}, \ldots, w_3^n, \overline{w_3^n}$ in this order and add the edges $w_1^j w_2^j, w_2^j w_3^j, \overline{w_1^j}\,\overline{w_2^j}, \overline{w_2^j}\,\overline{w_3^j}$ for every $j = 1, 2, \ldots, n$. Let $s = z_1^1$ and $t = z_2^n$.

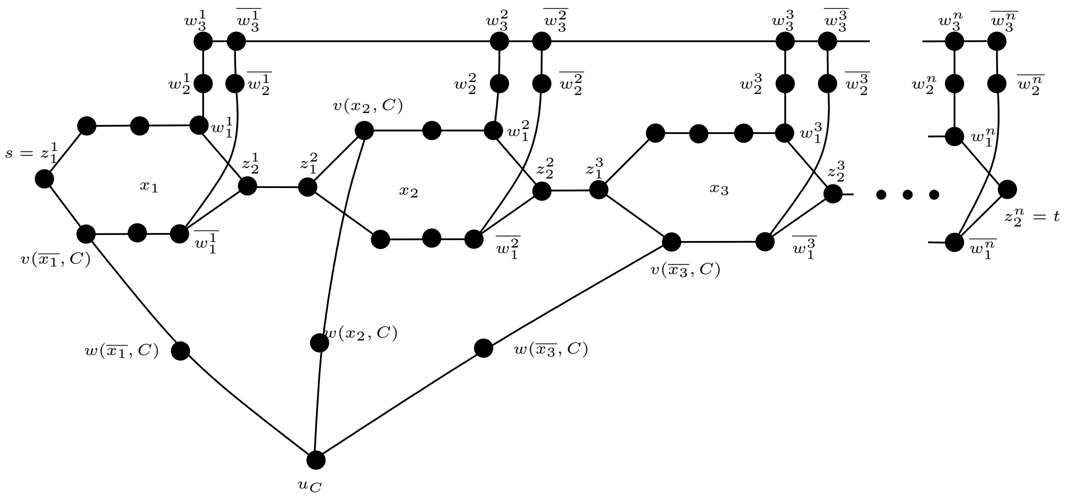

Figure 2: Part of the construction of graph G_φ for clause $C = (\overline{x_1} \vee x_2 \vee \overline{x_3})$.

The construction of the graph G_φ is finished. An illustration can be found in Figure 2.

Clearly, G_φ is simple and has maximum degree three.

If τ is a truth assignment to the variables $x_1, x_2, \ldots, x_n$ then we define an s-t path P_τ as follows: for every $j = 1, 2, \ldots, n$, if x_j is set to TRUE then let P_τ go through the nodes $z_1^j, v(\overline{x_j}, C_{b_1^j}), v(\overline{x_j}, C_{b_2^j}), \ldots, v(\overline{x_j}, C_{b_{l_j}^j})$, $\overline{w_1^j}, z_2^j$, otherwise (i.e., if x_j is set to FALSE) let P_τ go through $z_1^j, v(x_j, C_{a_1^j})$, $v(x_j, C_{a_2^j}), \ldots, v(x_j, C_{a_{k_j}^j}), w_1^j, z_2^j$.

We need one more concept. An s-t path P is called an *assignment-defining path* if $v \in V(P)$, $d_G(v) = 2$ implies $v \in \{s, t\}$. For such a path P we define the truth assignment τ_P such that $P_{\tau_P} = P$.

Claim 1. *There is an s-t path $P \subseteq E$ such that $(V, E-P)$ is connected if and only if $\varphi \in 3SAT$. Consequently, Problem* $\mathrm{P}_{st} \wedge \mathrm{SpT}$ *is NP-complete.*

Proof. If τ is a truth assignment showing that $\varphi \in 3SAT$ then P_τ is a path satisfying the requirements, as one can check. On the other hand, if P is an s-t path such that $(V,\ E-P)$ is connected then P cannot go through nodes of degree 2, therefore P is assignment-defining, and τ_P shows $\varphi \in 3SAT$. □

To show the NP-completeness of Problem T+SpT, modify G_φ the following way: subdivide the two edges incident to s with two new nodes s' and s'' and connect these two nodes with an edge. Repeat this with t: subdivide the two edges incident to t with two new nodes t' and t'' and connect t' and t''. Let the graph obtained this way be $G = (V, E)$. Clearly, G is subcubic and simple. Note that the definition of an assignment defining path and that of P_τ for a truth assignment τ can be obviously modified for the graph G.

Claim 2. *There exists a truth assignment τ such that every clause in φ contains exactly one true literal if and only if there exists a set $T \subseteq E$ such that $(V(T),\ T)$ is a tree and $(V,\ E–T)$ is a spanning tree. Consequently, Problem* T+SpT *is NP-complete.*

Proof. If τ is a truth assignment as above then one can see that $T = P_\tau$ is an edge set satisfying the requirements.

On the other hand, assume that $T \subseteq E$ is such that $(V(T),\ T)$ is a tree and $T^* = (V,\ E-T)$ is a spanning tree. Since T^* cannot contain circuits, T must contain at least one of the 3 edges $ss', s's'', s''s$ (call it e), as well as at least one of the 3 edges $tt', t't'', t''t$ (say f). Since $(V(T),\ T)$ is connected, T contains a path $P \subseteq T$ connecting e and f (note that since $(V, E-T)$ is connected, $|T \cap \{ss', s's'', s''s\}| = |T \cap \{tt', t't'', t''t\}| = 1$). Since $(V,\ E-P)$ is connected, P cannot go through nodes of degree 2 (except for the endnodes of P), and the edges e and f must be the last edges of P (otherwise P would disconnect s or t from the rest). Thus, without loss of generality we can assume that P connects s and t (by locally changing P at its ends), and we get that P is assignment defining. Observe that in fact T must be equal to P, since G is subcubic (therefore T cannot contain nodes of degree 3). Consider the truth assignment τ_P associated to P, we claim that τ_P satisfies our requirements. Clearly, if a clause C of φ does not contain a true literal then u_C is not reachable from s in $G-T$, therefore every clause of φ contains at least one true literal. On the other hand assume that a clause C contains at least 2 true literals (say x_j and $\overline{x_k}$ for some $j \neq k$), then one can see that there exists a circuit in $G-T$ (because $v(x_j, C)$ is still reachable from $v(\overline{x_k}, C)$ in $G-T-u_C$ via the nodes w_j^1, w_j^2, w_j^3 and $\overline{w_k^1}, \overline{w_k^2}, \overline{w_k^3}$). □

Finally we prove the NP-completeness of Problem C∧SpT. For the 3CNF formula φ with variables $x_1, x_2, \ldots, x_n$ and clauses $C_1, C_2, \ldots, C_m$, let us associate the 3CNF formula φ' with the same variable set and clauses $(x_1 \vee x_1 \vee \overline{x_1}),\ (x_2 \vee x_2 \vee \overline{x_2}), \ldots,\ (x_n \vee x_n \vee \overline{x_n}),\ C_1,\ C_2, \ldots, C_m$. Clearly, φ is satisfiable if and only if φ' is satisfiable. Construct the graph $G_{\varphi'} = (V, E)$ as above (the construction is clear even if some clauses contain only 2 literals), and let $G = (V, E)$ be obtained from $G_{\varphi'}$ by adding the edge st.

Claim 3. *The formula φ' is satisfiable if and only if there exists a set $K \subseteq E$ such that $(V(K),\ K)$ is a circuit and $G-K = (V,\ E-K)$ is connected. Consequently, Problem* C∧SpT *is NP-complete.*

Proof. First observe that if τ is a truth assignment satisfying φ' then $K = P_\tau \cup \{st\}$ is an edge set satisfying the requirements. On the other hand, if K is an edge set satisfying the requirements then K cannot contain nodes of degree 2, since $G-K$ is connected. We claim that K can neither be a circuit associated to a variable x_i, because in this case the node u_C associated to clause $C = (x_i \vee x_i \vee \overline{x_i})$ would not be reachable in $G-K$ from s. Therefore K consists of the edge st and an assignment defining path P. It is easy to check (analogously to the previous arguments) that τ_P is a truth assignment satisfying ϕ'. □

As we have proved the NP-completeness of all three problems, the theorem is proved. □

We note that the construction given in our original proof of the above theorem (see [5]) was used by Bang-Jensen and Yeo in [3]. They settled an open problem raised by Thomassé in 2005. They proved that it is NP-complete to decide SpA ∧ SpT in digraphs, where SpA denotes a spanning arborescence and SpT denotes a spanning tree in the underlying undirected graph.

We also point out that the planarity of the graphs in the above proofs cannot be assumed. We do not know the status of any of the Problems P_{st} ∧SpT, T+SpT, and T+T in planar graphs. It was shown in [4] that Problem C∧SpT is polynomially solvable in planar graphs. We also mention that planar duality gives that Problem C∧SpT in a planar graph is eqivalent to finding a cut in a planar graph that contains no circuit: by the results of [4], this problem is also polynomially solvable. However van den Heuvel [11] has shown that this problem is NP-complete for general (i.e., not necessarily planar) graphs.

We point out to an interesting connection towards the Graphic TSP Problem. This problem can be formulated as follows. Given a connected graph $G = (V, E)$, find a connected Eulerian subgraph of $2G$ spanning V with minimum number of edges (where $2G = (V, 2E)$ is the graph obtained from G by doubling its edges). The connection is the following. Assume that $F \subseteq 2E$ is a feasible solution to the problem. A greedy way of improving F would be to delete edges from it, while maintaining the feasibility. It is thus easy to observe that this greedy improvement is possible if and only if the graph (V, F) contains an edge-disjoint circuit and a spanning tree (which is Problem C$\wedge$SpT in our notations). However, slightly modifying the proof above it can be shown that Problem C$\wedge$SpT is also NP-complete in Eulerian graphs (details can be found in [4]).

Theorem 7. *Problem* Cut+F *is NP-complete.*

Proof. The problem is clearly in NP. In order to show its completeness let us first rephrase the problem. Given a graph, Problem Cut+F asks whether we can colour the nodes of this graph with two colours such that no monochromatic circuit exists.

Consider the NP-complete Problem 2-COLOURABILITY OF A 3-UNIFORM HYPERGRAPH This problem is the following: given a 3-uniform hypergraph $H = (V, \mathcal{E})$, can we colour the set V with two colours (say red and blue) such that there is no monochromatic hyperedge in $\mathcal{E}$ (the problem is indeed NP-complete, since Problem GT6 in [10] is a special case of this problem). Given the instance $H = (V, \mathcal{E})$ of this problem, construct the following graph G. The node set of G is $V \cup V_\mathcal{E}$, where $V_\mathcal{E}$ is disjoint from V and it contains 6 nodes for every hyperedge in $\mathcal{E}$: for an arbitrary hyperedge $e = \{v_1, v_2, v_3\} \in \mathcal{E}$, the 6 new nodes associated to it are $x_{v_1,e}, y_{v_1,e}, x_{v_2,e}, y_{v_2,e}, x_{v_3,e}, y_{v_3,e}$. The edge set of G contains the following edges: for the hyperedge $e = \{v_1, v_2, v_3\} \in \mathcal{E}$, v_i is connected with $x_{v_i,e}$ and $y_{v_i,e}$ for every $i = 1, 2, 3$, and among the 6 nodes associated to e every two is connected with an edge except for the 3 pairs of form $x_{v_i,e}, y_{v_i,e}$ for $i = 1, 2, 3$ (i.e., $|E(G)| = 18|\mathcal{E}|$). The construction of G is finished. An illustration can be found in Figure 3. Note that in any two-colouring of $V \cup V_\mathcal{E}$ the 6 nodes associated to the hyperedge $e = \{v_1, v_2, v_3\} \in \mathcal{E}$ do not induce a monochromatic circuit if and only if there exists a permutation i, j, k of $1, 2, 3$ so that they are coloured the following way: $x_{v_i,e}, y_{v_i,e}$ is blue, $x_{v_j,e}, y_{v_j,e}$ is red and $x_{v_k,e}, y_{v_k,e}$ are of different colour.

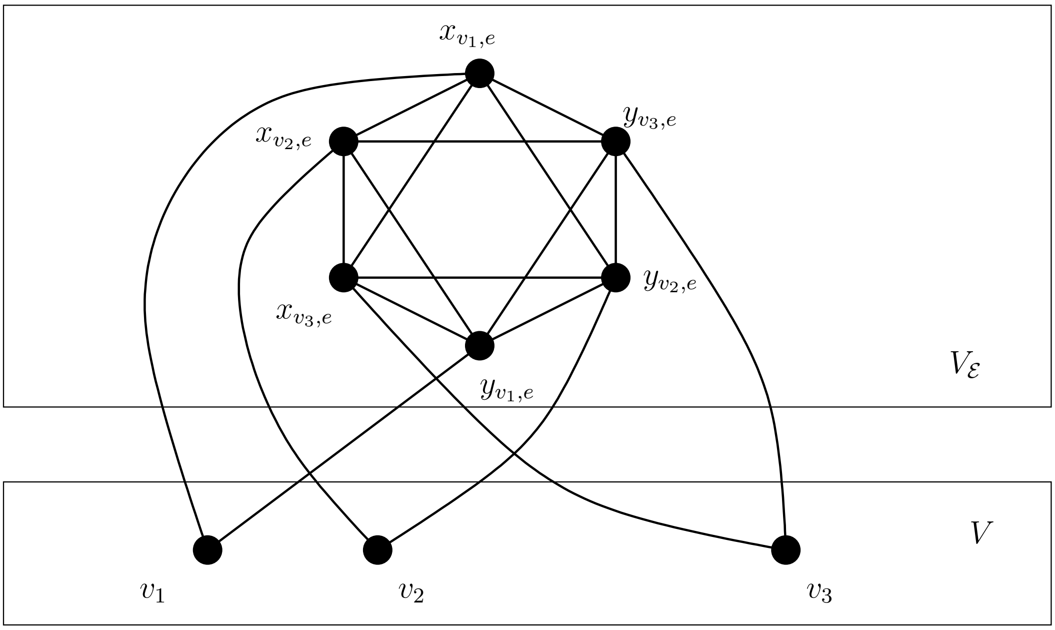

Figure 3: Part of the construction of the graph G in the proof of Theorem 7.

One can check that V can be coloured with 2 colours such that there is no monochromatic hyperedge in $\mathcal{E}$ if and only if $V \cup V_\mathcal{E}$ can be coloured with 2 colours such that there is no monochromatic circuit in G. $\square$

We note that we do not know the status of Problem Cut+F in planar graphs.

3 Algorithms

Algorithm for $\mathrm{P}_{st} \wedge \mathrm{C}$**.** Assume we are given a connected multigraph $G = (V, E)$ and two nodes $s, t \in V$, and we want to decide whether an s-t-path $P \subseteq E$ and a circuit $C \subseteq E$ exists with $P \cap C = \emptyset$. We may even assume that both s and t have degree at least two. If $v \in V - \{s, t\}$ has degree at most two then we can eliminate it. If there is a cut-vertex $v \in V$ then we can decompose the problem into smaller subproblems by checking whether s and t fall in the same component of $G-v$, or not. If they do then P should lie in that component, otherwise P has to go through v.

If there is a non-trivial two-edge s-t-cut (i.e., a set X with $\{s\} \subsetneq X \subsetneq V-t$, and $d_G(X) = 2$), then we can again reduce the problem in a similar way: the circuit to be found cannot use both edges entering X and we have to solve two smaller problems obtained by contracting X for the first one, and contracting $V-X$ for the second one.

So we can assume that $|E| \geq n + \lceil n/2 \rceil - 1$, and that G is 2-connected and G has no non-trivial two-edge s-t-cuts. Run a BFS from s and associate levels to vertices (s gets 0). If t has level at most $\lceil n/2 \rceil - 1$ then we have a path of length at most $\lceil n/2 \rceil - 1$ from s to t, after deleting its edges, at least n edges remain, so we are left with a circuit.

So we may assume that the level of t is at least $\lceil n/2 \rceil$. As G is 2-connected, we must have at least two vertices on each intermediate level. Consequently n is even, t is on level $n/2$, and we have exactly two vertices on each intermediate level, and each vertex $v \in V-\{s,t\}$ has degree 3, or, otherwise for a minimum s-t path P we have that $G-P$ has at least n edges, i.e., it contains a circuit. We have no non-trivial two-edge s-t-cuts, consequently there can only be two cases: either G equals to K_4 with edge st deleted, or G arises from a K_4 such that two opposite edges are subdivided (and these subdivision nodes are s and t). In either cases we have no solution.

Algorithm for $\mathrm{C} \wedge \mathrm{C}$. We give a simple polynomial time algorithm for deciding whether two edge-disjoint circuits can be found in a given connected multigraph $G = (V, E)$. We note that a polynomial (but less elegant) algorithm for this problem was also given in [6].

If any vertex has degree at most two, we can eliminate it, so we may assume that the minimum degree is at least 3. If G has at least 16 vertices, then it has a circuit of length at most $n/2$ (simply run a BFS from any node and observe that there must be a non-tree edge between some nodes of depth at most $\log(n)$, giving us a circuit of length at most $2\log(n) \leq n/2$), and after deleting the edges of this circuit, at least n edges remain, so we are left with another circuit. For smaller graphs we can check the problem in constant time.

4 Matroidal generalizations

In this section we will consider the matroidal generalizations for the problems that were shown to be polynomially solvable in the graphic matroid. In fact we will only need linear matroids, since it turns out that the problems we consider are already NP-complete in them. We will use the following result of Khachyan.

Theorem 8 (Khachyan [14])**.** *Given a $D \times N$ matrix over the rationals, it is NP-complete to decide whether there exist D linearly dependent columns.*

First we consider the matroidal generalization of Problem $\mathrm{Cut} \wedge \mathrm{C}$.

Theorem 9. *It is NP-complete to decide whether an (explicitly given) linear matroid contains a cut and a circuit that are disjoint.*

Proof. Observe that there is no disjoint cut and circuit in a matroid if and only if every circuit contains a base, that is equivalent with the matroid being uniform. Khachyan's Theorem 8 is equivalent with the uniformness of the linear matroid determined by the coloumns of the matrix in question, proving our theorem. □

Finally we consider the matroidal generalization of Problem $\mathrm{C} \wedge \mathrm{C}$ and $\mathrm{Cut} \wedge \mathrm{Cut}$.

Theorem 10. *The problem of deciding whether an (explicitly given) linear matroid contains two disjoint circuits is NP-complete.*

Proof. We will prove here that Khachyan's Theorem 8 is true even if $N = 2D + 1$, which implies our theorem, since there are two disjoint circuits in the linear matroid represented by this $D \times (2D+1)$ matrix if and only if there are D linearly dependent columns in it.

Khachyan's proof of Theorem 8 was simplified by Vardy [26], we will follow his line of proof. Consider the following problem.

Problem 11. *Given different positive integers $a_1, a_2, \ldots, a_n, b$ and a positive integer d, decide whether there exist d indices $1 \leq i_1 < i_2 < \cdots < i_d \leq n$ such that $b = a_{i_1} + a_{i_2} + \cdots + a_{i_d}$.*

Note that Problem 11 is very similar to the Subset-Sum Problem (Problem SP13 in [10]), the only difference being that in the Subset-Sum problem we do not specify d, and the numbers $a_1, a_2, \ldots, a_n$ need not be different. On the other hand, here we will strongly need that the numbers $a_1, a_2, \ldots, a_n$ are all different. Vardy has shown the following claim (we include a proof for sake of completeness).

Claim 4. *There is solution to Problem 11 if and only if there are $d+1$ linearly dependent columns (above the rationals) in the $(d+1)\times(n+1)$ matrix*

$$\begin{pmatrix} 1 & 1 & \cdots & 1 & 0 \\ a_1 & a_2 & \cdots & a_n & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ a_1^{d-2} & a_2^{d-2} & \cdots & a_n^{d-2} & 0 \\ a_1^{d-1} & a_2^{d-1} & \cdots & a_n^{d-1} & 1 \\ a_1^{d} & a_2^{d} & \cdots & a_n^{d} & b \end{pmatrix}.$$

Proof. We use the following facts about determinants. Given real numbers $x_1, x_2, \ldots, x_k$, we have the following well-known relation for the Vandermonde determinant:

$$\det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_k \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{k-1} & x_2^{k-1} & \cdots & x_k^{k-1} \end{pmatrix} = \prod_{i<j}(x_j - x_i).$$

Therefore the Vandermonde determinant is not zero, if the numbers $x_1, x_2, \ldots, x_k$ are different. Furthermore, we have the following relation for an alternant of the Vandermonde determinant (see Chapter V in [17], for example):

$$\det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_k \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{k-2} & x_2^{k-2} & \cdots & x_k^{k-2} \\ x_1^{k} & x_2^{k} & \cdots & x_k^{k} \end{pmatrix} = (x_1 + x_2 + \cdots + x_k)\prod_{i<j}(x_j - x_i).$$

W include a proof of this last fact: given an arbitrary $k \times k$ matrix $X = ((x_{ij}))$ and numbers $u_1, \ldots, u_k$, observe (by checking the coefficients of the u_is on each side) that

$$\det \begin{pmatrix} u_1x_{11} & u_2x_{12} & \cdots & u_kx_{1k} \\ x_{21} & x_{22} & \cdots & x_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ x_{k1} & x_{k2} & \cdots & x_{kk} \end{pmatrix} + \det \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1k} \\ u_1x_{21} & u_2x_{22} & \cdots & u_kx_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ x_{k1} & x_{k2} & \cdots & x_{kk} \end{pmatrix} + \cdots +$$

$$\det \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1k} \\ x_{21} & x_{22} & \cdots & x_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ u_1x_{k1} & u_2x_{k2} & \cdots & u_kx_{kk} \end{pmatrix} = (u_1 + u_2 + \ldots u_k)\det \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1k} \\ x_{21} & x_{22} & \cdots & x_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ x_{k1} & x_{k2} & \cdots & x_{kk} \end{pmatrix}$$

Now apply this to the Vandermonde matrix $X = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_k \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{k-1} & x_2^{k-1} & \cdots & x_k^{k-1} \end{pmatrix}$ and numbers $u_i = x_i$ for every $i = 1, 2, \ldots, k$.

We will use these two facts. The first one implies that if $d+1$ columns of our matrix are dependent then they have to include the last column. By the second fact, if $1 \le i_1 < i_2 < \cdots < i_d \le n$ are arbitrary indices then

$$\det \begin{pmatrix} 1 & 1 & \cdots & 1 & 0 \\ a_{i_1} & a_{i_2} & \cdots & a_{i_d} & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ a_{i_1}^{d-2} & a_{i_2}^{d-2} & \cdots & a_{i_d}^{d-2} & 0 \\ a_{i_1}^{d-1} & a_{i_2}^{d-1} & \cdots & a_{i_d}^{d-1} & 1 \\ a_{i_1}^{d} & a_{i_2}^{d} & \cdots & a_{i_d}^{d} & b \end{pmatrix} = (b - a_{i_1} - a_{i_2} - \cdots - a_{i_d})\prod_{k<l}(a_{i_l} - a_{i_k}).$$

This implies the claim. □

Vardy also claimed that Problem 11 is NP-complete: our proof will be completed if we show that this is indeed the case even if $n = 2d+2$. Since we have not found a formal proof of this claim of Vardy, we will give a full proof of the following claim. For a set V let $\binom{V}{3} = \{X \subseteq V : |X| = 3\}$.

Claim 5. *Problem 11 is NP-complete even if $n = 2d + 2$.*

Proof. We will reduce the well-known NP-complete problem Exact-Cover-by-3-Sets (Problem SP2 in [10]) to this problem. Problem Exact-Cover-by-3-Sets is the following: given a 3-uniform family $\mathcal{E} \subseteq \binom{V}{3}$, decide whether there exists a subfamily $\mathcal{E}' \subseteq \mathcal{E}$ such that every element of V is contained in exactly one member of $\mathcal{E}'$. We assume that 3 divides $|V|$, and let $d = |V|/3$, so Problem Exact-Cover-by-3-Sets asks whether there exist d disjoint members in $\mathcal{E}$. First we show that this problem remains NP-complete even if $|\mathcal{E}| = 2d + 2$. Indeed, if $|\mathcal{E}| \neq 2d + 2$ then let us introduce $3k$ new nodes $\{u_i, v_i, w_i : i = 1, 2, \ldots, k\}$ where

- k is such that $\binom{3k}{3} - 2k \geq 2d + 2 - |\mathcal{E}|$ if $|\mathcal{E}| < 2d + 2$, and
- $k = |\mathcal{E}| - (2d + 2)$, if $|\mathcal{E}| > 2d + 2$.

Let $V^* = V \cup \{u_i, v_i, w_i : i = 1, 2, \ldots, k\}$ and let $\mathcal{E}^* = \mathcal{E} \cup \{\{u_i, v_i, w_i\} : i = 1, 2, \ldots, k\}$ (note that $|V^*| = 3(d + k)$). If $|\mathcal{E}| < 2d + 2$ then include furthermore $2(d + k) + 2 - (|\mathcal{E}| + k)$ arbitrary new sets of size 3 to $\mathcal{E}^*$ from $\binom{V^* - V}{3}$, but so that $\mathcal{E}^*$ does not contain a set twice (this can be done by the choice of k). It is easy to see that $|\mathcal{E}^*| = 2|V^*|/3 + 2$, and V can be covered by disjoint members of $\mathcal{E}$ if and only if V^* can be covered by disjoint members of $\mathcal{E}^*$.

Finally we show that Exact-Cover-by-3-Sets is a special case of Problem 11 in disguise. Given an instance of Exact-Cover-by-3-Sets by a 3-uniform family $\mathcal{E} \subseteq \binom{V}{3}$, consider the characteristic vectors of these 3-sets as different positive integers written in base 2 (that is, assume a fixed ordering of the set V, then the characteristic vectors of the members in $\mathcal{E}$ are 0-1 vectors corresponding to different binary integers, conatining 3 ones in their representation). These will be the numbers $a_1, a_2, \ldots, a_{|\mathcal{E}|}$. Let $b = 2^{|V|} - 1$ be the number corresponding to the all 1 characteristic vector, and let $d = |V|/3$. Observe that there exist d disjoint members in $\mathcal{E}$ if and only if there are indices $1 \leq i_1 < i_2 < \cdots < i_d \leq |\mathcal{E}|$ such that $b = a_{i_1} + a_{i_2} + \cdots + a_{i_d}$. (You will need to prove a small claim about the maximal number of ones in the binary representation of a sum of positive integers.) This together with the previous observation proves our claim. □

By combining Claims 4 and 5 we obtain the proof of Theorem 10 as follows. Consider an instance of Problem 11 with $n = 2d + 2$ and let $D = d + 1$. Claim 4 states that this instance has a solution if and only if the $(d+1) \times (n+1) = D \times (2D + 1)$ matrix defined in the claim has D linearly dependent columns, which must be NP-hard to decide by Claim 5. □

Corollary 12. *The problem of deciding whether an (explicitly given) linear matroid contains two disjoint cuts is NP-complete.*

Proof. Since the dual matroid of the linear matroid is also linear, and we can construct a representation of this dual matroid from the representation of the original matroid, this problem is equivalent to the problem of deciding whether a linear matroid contains two disjoint circuits, which is NP-complete by Theorem 10. □

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